

LECTURE NOTES ON COMPLEX INTERPOLATION OF COMPACTNESS - PRELIMINARY VERSION

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ABSTRACT. We recall that the fundamental theorem of complex interpolation is the

Boundedness Theorem: If, for $j = 0, 1$, a linear operator T is a bounded map from the Banach space X_j to the Banach space Y_j then, for each θ , $0 < \theta < 1$, T is a bounded map between the complex interpolation spaces $[X_0, X_1]_\theta$ and $[Y_0, Y_1]_\theta$.

Alberto Calderón, in his foundational presentation of this material fifty-one years ago [4], also proved the following companion result:

Compactness Theorem/Question: Furthermore *in some cases*, if T is also a compact map from X_0 to Y_0 , then, for each θ , T is a compact map from $[X_0, X_1]_\theta$ to $[Y_0, Y_1]_\theta$.

The fundamental question of exactly which cases could be covered by such a result was not resolved then, and is still not resolved. In a previous paper [10] we surveyed several of the partial answers which have been obtained to this question, with particular emphasis on the work of Nigel Kalton in a joint paper [8] with one of us. This is a preliminary version of a set of lecture notes which will be a sequel to [10]. In them, for the most part, we will amplify upon various technical details of the contents of [8]. For example we plan to give a more explicit explanation of why the positive answer in [8] to the above question when (X_0, X_1) is a couple of lattices holds without any restriction on those lattices, and we also plan to provide more detailed versions of some of the other proofs in that paper. The main purpose of this preliminary version is to present two apparently new small results, pointing out a previously unnoticed particular case where the answer to the above question is affirmative. As our title suggests, this and future versions of these notes are intended to be more accessible to graduate students than a usual research article.

1. INTRODUCTION

This is a preliminary version of a set of lecture notes which will be a mostly technical sequel to [10], only intended for readers who are familiar with the contents of [10], or at least with Sections 1 to 3 of that paper. We shall unhesitatingly use any and all of the notation, terminology and conventions introduced there, usually without further explanation. This means, among other things, that all Banach spaces considered here will be over the complex field. It will also be necessary to consult a number of other references. In particular, we will assume that the reader is familiar with the definitions and various properties of the complex interpolation spaces $[A_0, A_1]_\theta$ and $[A_0, A_1]^\theta$ and the auxiliary spaces $\mathcal{F}(A_0, A_1)$ and $\overline{\mathcal{F}}(A_0, A_1)$ of $A_0 + A_1$ valued functions on the strip $\mathbb{S} = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ which are used in their construction, as can be found in the earlier sections of [4] or (with slightly different notation) in Chapter 4 of [2].

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Our main goal will be to provide some extra details about some tools which might ultimately be relevant for helping to answer a 51 year old question, which, as in [10], we will refer to as *Question CIC*. So far, those who have considered this question have had to content themselves with finding various special cases in which its answer is affirmative. I.e., rather than determining whether or not $(*, *) \blacktriangleright (*, *)$ holds, they have merely found various examples, sometimes quite large families of examples, of complex Banach couples (X_0, X_1) and (Y_0, Y_1) which satisfy $(X_0, X_1) \blacktriangleright (Y_0, Y_1)$.

We have deliberately used rather exotic notation in the preceding paragraph, in order to further emphasize that we assume familiarity with [10], where that notation is explained, and where some history of this question is recalled.

Given the fact that Question CIC has remained open for five decades, together with the fact that in the nineteen years since the publication of [8] there has been rather little further progress towards answering it, we feel that there is a case for carefully looking again at the details of [8]. But we have deferred doing this until a later version of these notes.

Our own recent rereading of [8] has born some modest fruit, by prompting us to discover at least one new family of Banach couples (B_0, B_1) for which $(*, *) \blacktriangleright (B_0, B_1)$. Since we are impatient to report some progress, even if rather small, in the battle with Question CIC, we have prepared this preliminary version of our notes mainly for this purpose.

Although we have by and large adopted a format which we hope will more conveniently accessible to and useful for graduate students, we should hasten to add that we most certainly do *not* wish to encourage anyone beginning a mathematical career to choose answering Question CIC as the main topic for her or his dissertation. That would really be a “high risk trajectory”.

We originally wrote [10] as the fourth of a series of papers intended to describe some small part of the most impressive body of mathematical research created by our brilliant colleague and dear friend Nigel Kalton. The first three of these papers were coauthored with Mario Milman. All of them have been posted on the arXiv. We have submitted material taken from those papers to also possibly appear in a “Selecta” volume to be published in Nigel’s memory.

2. A NEW RESULT VIA SCHAUDER’S THEOREM

In this section, in Corollary 2.5, we find, as promised, one more class of Banach couples (B_0, B_1) for which $(*, *) \blacktriangleright (B_0, B_1)$, i.e., for which $(A_0, A_1) \blacktriangleright (B_0, B_1)$ for all Banach couples (A_0, A_1) . The result is not entirely surprising, in view of Theorem 9 of [8, p. 271] and Schauder’s classical theorem about adjoints of compact operators. Indeed our result emerges as a consequence (see Theorem 2.4) of a Schauder-like theorem for complex interpolation of compact operators.

It will be convenient (as in [6, p. 22]) to use the notation $T : (A_0, A_1) \xrightarrow{c,b} (B_0, B_1)$, to mean that (A_0, A_1) and (B_0, B_1) are Banach couples and that $T : A_0 + A_1 \rightarrow B_0 + B_1$ is a linear operator which maps A_0 compactly into B_0 and A_1 boundedly into B_1 .

To begin our discussion we take the liberty of recalling an obvious fact which has been frequently used, in [8] and elsewhere:

Fact 2.1. *Let (A_0, A_1) and (B_0, B_1) be Banach couples, and let (X_0, X_1) and (Y_0, Y_1) be the regular Banach couples obtained by letting X_j be the closure of $A_0 \cap A_1$ in A_j and*

Y_j be the closure of B_j in $B_0 \cap B_1$ for $j = 0, 1$. Then $(X_0, X_1) \blacktriangleright (Y_0, Y_1)$ implies that $(A_0, A_1) \blacktriangleright (B_0, B_1)$.

We also take the liberty of recalling that Fact 2.1 is an immediate consequence of the obvious fact that any linear operator T satisfying $T : (A_0, A_1) \xrightarrow{c,b} (B_0, B_1)$ also satisfies $T : (X_0, X_1) \xrightarrow{c,b} (Y_0, Y_1)$ and also the very well known fact (see Section 9.3 of [4, p. 116]) that $[A_0, A_1]_\theta = [X_0, X_1]_\theta$ isometrically (and analogously $[B_0, B_1]_\theta = [Y_0, Y_1]_\theta$ isometrically). (Incidentally, since any linear operator satisfying $T : (X_0, X_1) \xrightarrow{c,b} (Y_0, Y_1)$ can be uniquely extended to be an operator \tilde{T} satisfying $\tilde{T} : (A_0, A_1) \xrightarrow{c,b} (B_0, B_1)$, we can also observe that the two properties $(X_0, X_1) \blacktriangleright (Y_0, Y_1)$ and $(A_0, A_1) \blacktriangleright (B_0, B_1)$ are in fact equivalent.)

Our discussion continues with the following auxiliary result:

Theorem 2.2. *Let (A_0, A_1) and (B_0, B_1) be arbitrary Banach couples. Let U be a bounded linear operator from $A_0 + A_1$ into $B_0 + B_1$. Suppose that, for some $\theta \in (0, 1)$, the operator U maps $[A_0, A_1]_\theta$ compactly into $[B_0, B_1]_\theta$. Then U also maps $[A_0, A_1]^\theta$ compactly into $[B_0, B_1]_\theta$.*

Remark 2.3. The boundedness of $U : A_0 + A_1 \rightarrow B_0 + B_1$ will of course hold whenever the much stronger property $U : (A_0, A_1) \xrightarrow{c,b} (B_0, B_1)$ is assumed to hold. We also remark that the conclusion of the theorem also holds if $[A_0, A_1]^\theta$ is replaced by the (possibly larger?) Gagliardo closure¹ in $A_0 + A_1$ of $[A_0, A_1]^\theta$. The following proof has some similarities with the proof of (b) in the second paragraph of the proof of Corollary 7 of [8, p. 270].

Proof. Let a be an arbitrary element of $[A_0, A_1]^\theta$. There exists a function $f \in \overline{\mathcal{F}}(A_0, A_1)$ for which $a = f'(\theta)$. Since f is an analytic $A_0 + A_1$ valued function in the open strip \mathbb{S}° , this means that $\lim_{h \rightarrow 0} \ell \left(\frac{f(\theta+h) - f(\theta)}{h} - f'(\theta) \right) = 0$ for every bounded linear functional ℓ on $A_0 + A_1$. In turn this implies that

$$(2.1) \quad \lim_{h \rightarrow 0} \lambda \left(U \left(\frac{f(\theta+h) - f(\theta)}{h} \right) - U f'(\theta) \right) = 0$$

for every bounded linear functional λ on $B_0 + B_1$. It is obvious from the definition² of $\overline{\mathcal{F}}(A_0, A_1)$ (as already observed and used long ago in [4, p. 136] and also used elsewhere, e.g., in [5, p. 1006] and [7]) that, for each $n \in \mathbb{N}$, the function $f_n : \mathbb{S} \rightarrow A_0 + A_1$ defined by $f_n(z) = ine^{z^2/n} (f(z + 1/in) - f(z))$ is an element of $\mathcal{F}(A_0, A_1)$ and satisfies $\|f_n\|_{\mathcal{F}(A_0, A_1)} \leq e^{1/n} \|f\|_{\overline{\mathcal{F}}(A_0, A_1)}$. It follows that $\{f_n(\theta)\}_{n \in \mathbb{N}}$ is a bounded sequence in $[A_0, A_1]_\theta$ and therefore $\{U f_n(\theta)\}_{n \in \mathbb{N}}$ has a subsequence which converges in $[B_0, B_1]_\theta$ norm to an element $b \in [B_0, B_1]_\theta$. This convergence must also occur with respect to the norm of $B_0 + B_1$ (since of course $[B_0, B_1]_\theta$ is continuously embedded in $B_0 + B_1$), and therefore it also occurs with respect to the weak topology of $B_0 + B_1$. Since the limit of a weakly convergent sequence is unique, we deduce, using (2.1), that b must equal $U f'(\theta) = Ua$.

¹This is the space whose unit ball is the closure of the unit ball of $[A_0, A_1]_\theta$ with respect to the norm of $A_0 + A_1$.

²Note that in [2] the notation $\mathcal{G}(A_0, A_1)$ is used to denote the space $\overline{\mathcal{F}}(A_0, A_1)$ of [4]. The same notation $\mathcal{G}(A_0, A_1)$ has a quite different meaning in [4].

Now let $\{a_k\}_{k \in \mathbb{N}}$ be an arbitrary bounded sequence in $[A_0, A_1]^\theta$. The arguments of the preceding paragraph imply, for each $k \in \mathbb{N}$, that $Ua_k \in [B_0, B_1]_\theta$ and also there exists an element $x_k \in [A_0, A_1]_\theta$ such that $\|x_k\|_{[A_0, A_1]_\theta} \leq e \|a_k\|_{[A_0, A_1]^\theta}$ and $\|Ua_k - Ux_k\|_{[B_0, B_1]_\theta} \leq 1/k$. The fact that $U : [A_0, A_1]_\theta \rightarrow [B_0, B_1]_\theta$ is compact ensures that $\{Ux_k\}_{k \in \mathbb{N}}$ has a convergent subsequence in $[B_0, B_1]_\theta$ and therefore that the same is true of $\{Ua_k\}_{k \in \mathbb{N}}$. \square

Now we can state the main results of this section.

Theorem 2.4. *Let (A_0, A_1) and (B_0, B_1) be Banach couples, and let (X_0, X_1) and (Y_0, Y_1) be the regular Banach couples obtained by letting X_j be the closure of $A_0 \cap A_1$ in A_j and Y_j be the closure of B_j in $B_0 \cap B_1$ for $j = 0, 1$. Suppose that the dual couples (X_0^*, X_1^*) and (Y_0^*, Y_1^*) satisfy $(Y_0^*, Y_1^*) \blacktriangleright (X_0^*, X_1^*)$. Then $(A_0, A_1) \blacktriangleright (B_0, B_1)$.*

Corollary 2.5. *Let (A_0, A_1) and (B_0, B_1) be Banach couples, such that B_0 is a UMD-space. Then $(A_0, A_1) \blacktriangleright (B_0, B_1)$.*

Remark 2.6. It seems natural to conjecture that a sort of converse to Theorem 2.4 also holds, namely that $(A_0, A_1) \blacktriangleright (B_0, B_1)$ or, equivalently, $(X_0, X_1) \blacktriangleright (Y_0, Y_1)$ is sufficient to imply that $(Y_0^*, Y_1^*) \blacktriangleright (X_0^*, X_1^*)$. However, as explained in the second version of [10], proving this conjecture, even for just one special particular choice of (X_0, X_1) and (Y_0, Y_1) , namely $(X_0, X_1) = (\ell^1(FL^1), \ell^1(FL_1^1))$ and $(Y_0, Y_1) = (\ell^\infty(FL^\infty), \ell^\infty(FL_1^\infty))$, would be equivalent to obtaining a positive answer for Question CIC.

Remark 2.7. In the proof of Corollary 2.5 we will need to use two fairly obvious facts, namely that the dual Y^* of a UMD-space Y and also every closed subspace of Y are also both UMD-spaces. These are mentioned without proof in several papers (including [8]). Formal statements of these facts can be found as parts (v) and (viii) respectively of Theorem 4.5.2 of [1, p. 145] and the proof of (v) there is provided as one of the consequences of Theorem 4.3.6 on p. 139 of the same book. It seems that there are easier settings in which to write such proofs, for example in the context of Corollary 2.18 of [3, p. 495], if one uses the characterization of UMD-spaces which appears there. In the future full version of these notes we may perhaps offer a fairly “self contained” treatment of the material that we require about UMD-spaces, working merely in terms of trigonometric polynomials and thus bypassing the need for various technical details.

Proof of Corollary 2.5. Let (X_0, X_1) and (Y_0, Y_1) be the regular Banach couples obtained from (A_0, A_1) and (B_0, B_1) as in the statement of Theorem 2.4. Since B_0 is a UMD-space, so is its closed subspace Y_0 and therefore so is the dual Y_0^* of Y_0 . Accordingly, Theorem 9 of [8, p. 271] implies that $(Y_0^*, Y_1^*) \blacktriangleright (X_0^*, X_1^*)$. The desired conclusion now follows from Theorem 2.4. \square

Proof of Theorem 2.4. We will present this proof using the somewhat pedantic language and notation of [7]. The main ingredient of the proof is Schauder’s theorem about the compactness of the adjoint of a linear operator. We will use the variant of that theorem which appears as Theorem 2.7 of [6, p. 21]. In preparation for that, as in Section 1 of [7], we shall introduce the bilinear functional $\langle \cdot, \cdot \rangle$ defined on $(X_0 \cap X_1) \times (X_0 \cap X_1)^*$ which defines the duality between $X_0 \cap X_1$ and its dual. Then, for each regular intermediate space X with respect to the couple (X_0, X_1) , we define the space $X^\#$ as in Definition 1.4 of [7, p. 3].

The Hahn-Banach Theorem, together with the fact ([7, Fact 1.6, p. 4]) that $X^\#$ identifies isometrically with the dual X^* of X imply that

$$(2.2) \quad \|x\|_X = \sup \{ |\langle x, z \rangle| : z \in X^\#, \|z\|_{X^\#} \leq 1 \} \text{ for all } x \in X_0 \cap X_1.$$

In particular we will be considering the cases where X is X_0 or X_1 or $[X_0, X_1]_\theta$. We will also be using the fact that

$$(2.3) \quad ([X_0, X_1]_\theta)^\# = [X_0^\#, X_1^\#]^\theta \text{ with equality of norms}$$

which, since $X_0 \cap X_1$ is dense in $[X_0, X_1]_\theta$, is just a reformulation of Calderón's duality theorem ([4] Section 12.1 p. 121 and Section 32.1 pp. 148–156, or [2, pp. 98–101], or see an alternative proof in Section 2 of [7]).

We will need the exact analogues of these notions for the couple (Y_0, Y_1) . To avoid confusion we shall use a different notation, namely $\langle\langle \cdot, \cdot \rangle\rangle$ for the bilinear functional defined on $(Y_0 \cap Y_1) \times (Y_0 \cap Y_1)^*$ which defines the duality between $Y_0 \cap Y_1$ and its dual. For each regular intermediate space Y with respect to the couple (Y_0, Y_1) we can unambiguously use the notation $Y^\#$ for the space defined, as in Definition 1.4 of [7, p. 3] (but of course using $\langle\langle \cdot, \cdot \rangle\rangle$). Here again we will particularly need to consider the cases where Y is Y_0 , Y_1 or $[Y_0, Y_1]_\theta$.

Let $T : (X_0, X_1) \rightarrow (Y_0, Y_1)$ be a linear operator satisfying $\|T\|_{X_j \rightarrow Y_j} \leq 1$ for $j = 0, 1$. We shall define the function $h : (X_0 \cap X_1) \times (Y_0 \cap Y_1)^* \rightarrow \mathbb{C}$ by setting

$$(2.4) \quad h(a, b) = \langle\langle Ta, b \rangle\rangle \text{ for all } a \in X_0 \cap X_1 \text{ and all } b \in (Y_0 \cap Y_1)^*$$

For each fixed $b \in (Y_0 \cap Y_1)^*$ let Sb be the linear functional on $X_0 \cap X_1$ which is defined by $Sb(a) = h(a, b)$ for each $a \in X_0 \cap X_1$. Then

$$|Sb(a)| \leq \|T\|_{X_0 \cap X_1 \rightarrow Y_0 \cap Y_1} \|a\|_{X_0 \cap X_1} \|b\|_{(Y_0 \cap Y_1)^*} \leq \|a\|_{X_0 \cap X_1} \|b\|_{(Y_0 \cap Y_1)^*}$$

which shows that $S : (Y_0 \cap Y_1)^* \rightarrow (X_0 \cap X_1)^*$ with $\|S\|_{(Y_0 \cap Y_1)^* \rightarrow (X_0 \cap X_1)^*} \leq 1$. Thus we can rewrite $Sb(a)$ as $\langle a, Sb \rangle$.

Remark 2.8. Of course S is essentially the adjoint of T . But we need to apply the operators T and S to several different spaces, i.e., it would seem that the bilinear functionals expressing the relevant dualities for these different spaces are defined on different spaces. Hence our preference to proceed cautiously, and to use the pedantic notation of [7]. We should perhaps explicitly recall that, as remarked after Definition 1.4 of [7, p. 3], $(X_0 \cap X_1)^* = (X_0 \cap X_1)^\#$, and furthermore $(X_0 \cap X_1)^\# = X_0^\# + X_1^\#$ (cf. [7, Fact 1.9, p. 5]), so that we have $S : Y_0^\# + Y_1^\# \rightarrow X_0^\# + X_1^\#$.

Now let X and Y be regular intermediate spaces with respect to (X_0, X_1) and (Y_0, Y_1) respectively, and suppose that they are also relative exact interpolation spaces with respect to these couples. Then $T : X \rightarrow Y$ with $\|T\|_{X \rightarrow Y} \leq 1$. It is an easily checked and essentially standard fact that

$$(2.5) \quad S : Y^\# \rightarrow X^\# \text{ with } \|S\|_{Y^\# \rightarrow X^\#} \leq 1.$$

Let us nevertheless recall the argument which gives (2.5). Recall that $Y^\#$ is the subspace of $(Y_0 \cap Y_1)^*$ consisting of those elements $y^* \in (Y_0 \cap Y_1)^*$ which satisfy

$$\|y^*\|_{Y^\#} := \sup \{ |\langle y, y^* \rangle| : y \in Y_0 \cap Y_1, \|y\|_Y \leq 1 \} < \infty.$$

Remark 2.9. Note that, in particular, $(Y_0 \cap Y_1)^\# \stackrel{1}{=} (Y_0 \cap Y_1)^*$. In fact what we are doing here (and in [7]) is consistent with the standard procedure used in many papers (and thus also implicitly in the statement of Theorem 2.4) of defining the dual couple (Y_0^*, Y_1^*) of a regular couple (Y_0, Y_1) by identifying Y_0^* and Y_1^* as spaces which are continuously embedded into $(Y_0 \cap Y_1)^*$. Therefore, if we use that identification, we have $Y_0^* = Y_0^\#$ and $Y_1^* = Y_1^\#$ with equality of norms.

Continuing our verification of (2.5), we observe that, since $Y_0 \cap Y_1$ is continuously embedded in Y , it follows that $Y^\#$ is continuously embedded in $(Y_0 \cap Y_1)^*$. Thus Sy^* is a defined element of $(X_0 \cap X_1)^*$ for each $y^* \in Y^\#$ and, for each $x \in X_0 \cap X_1$, we have $Tx \in Y_0 \cap Y_1$ and

$$\begin{aligned} |Sy^*(x)| &= |\langle x, Sy^* \rangle| = |\langle Tx, y^* \rangle| \leq \|Tx\|_Y \|y^*\|_{Y^\#} \\ &\leq \|x\|_X \|y^*\|_{Y^\#}. \end{aligned}$$

This means exactly that $Sy^* \in X^\#$ and $\|Sy^*\|_{X^\#} \leq \|y^*\|_{Y^\#}$. Thus we have established (2.5).

For X and Y having the properties specified above, we now let $A = X_0 \cap X_1 \cap \mathcal{B}_X$ and $B = (Y_0 \cap Y_1)^* \cap \mathcal{B}_{Y^\#} = \mathcal{B}_{Y^\#}$. (Here, in order to avoid confusion with the set B we are denoting the closed unit ball of any Banach space E by \mathcal{B}_E rather than by B_E .) Consider the semimetric spaces (A, d_A) and (B, d_B) defined exactly as in Theorem 2.7 of [6, p. 21], here using the function h defined, as above, by (2.4), now restricted of course to $A \times B$. For this choice of A , B and h it follows that, for each a_1 and a_2 in A , using the analogue of (2.2) for the couple (Y_0, Y_1) , we have

$$\begin{aligned} d_A(a_1, a_2) &= \sup \{ |\langle Ta_1 - Ta_2, b \rangle| : b \in \mathcal{B}_{Y^\#} \} \\ &= \|Ta_1 - Ta_2\|_Y. \end{aligned}$$

Furthermore, for each b_1 and b_2 in B , in view of the definition of $X^\#$ and of its norm,

$$\begin{aligned} d_B(b_1, b_2) &= \sup \{ |\langle a, Sb_1 - Sb_2 \rangle| : a \in \mathcal{B}_X \cap X_0 \cap X_1 \} \\ &= \|Sb_1 - Sb_2\|_{X^\#}. \end{aligned}$$

The formula for d_B shows that (B, d_B) is totally bounded if and only if $S : Y^\# \rightarrow X^\#$ is compact. Since A is dense in \mathcal{B}_X , the formula for d_A shows that (A, d_A) is totally bounded if and only if $T : X \rightarrow Y$ is compact. Thus, Theorem 2.7 of [6, p. 21] gives us the Schauder type result that

$$(2.6) \quad S : Y^\# \rightarrow X^\# \text{ is compact if and only if } T : X \rightarrow Y \text{ is compact.}$$

(Let us quickly mention that various quantitative versions of (2.6) expressed in terms of covering numbers, can be obtained by applying quantitative versions of [6, Theorem 2.7] which appear in [9] or other results referred to on the first page of [9].)

Now suppose that $T : X_0 \rightarrow Y_0$ is compact, in addition to the other conditions which T was already assumed to satisfy. Then $S : Y_0^\# \rightarrow X_0^\#$ is compact. The fact that $(Y_0^\#, Y_1^\#) \blacktriangleright$

$(X_0^\#, X_1^\#)$ then tells us that $S : [Y_0^\#, Y_1^\#]_\theta \rightarrow [X_0^\#, X_1^\#]_\theta$ is compact. We can now use Theorem 2.2 together with the fact (see Section 8 of [4, p. 116]) that $[X_0^\#, X_1^\#]_\theta$ is continuously embedded into $[X_0^\#, X_1^\#]^\theta$ to obtain that $S : [Y_0^\#, Y_1^\#]^\theta \rightarrow [X_0^\#, X_1^\#]^\theta$ is compact. This, combined with (2.3) (and of course its counterpart for (Y_0, Y_1)) and with (2.6) enables us to obtain that $T : [X_0, X_1]_\theta \rightarrow [Y_0, Y_1]_\theta$ is compact. This shows that $(X_0, X_1) \blacktriangleright (Y_0, Y_1)$ and therefore (cf. Fact 2.1) completes the proof of the theorem. \square

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